Comments on Opfer's alleged proof of the $3n + 1$ Conjecture

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The $3n + 1$ *map* $T : \mathbb{N} \to \mathbb{N}$ is defined by $T(n) = \dfrac{n}{2}$ if n is even, while $T(n) = \dfrac{3n+1}{2}$ if n is odd. The $3n + 1$ *graph* is the directed graph with the natural numbers as vertices, and the edges are from n to $T(n)$ for each $n \in \mathbb{N}$. Figure 1 shows the part of the graph containing the vertices for $n \le 100$. The $3n+1$ *conjecture*, also called *Collatz conjecture*, states that for every $n \in \mathbb{N}$ the directed path in the $3n+1$ graph starting at n will eventually reach 1. The $3n + 1$ conjecture is equivalent to the statement that the $3n + 1$ graph is weakly connected.

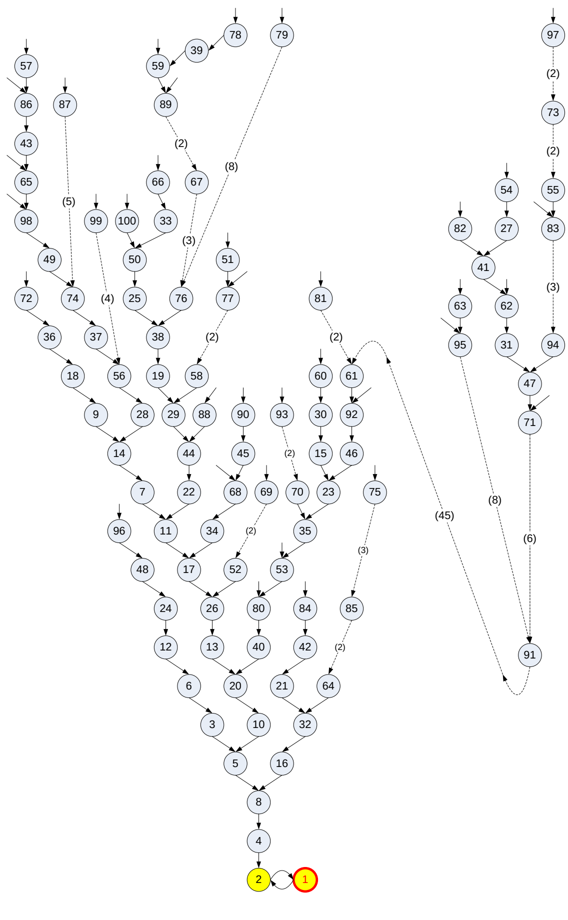

Figure 1. The $3n + 1$-graph.

Gerhard Opfer [O] claims to have proved the $3n + 1$ conjecture. The present note provides some comments on the arguments presented in [O].

In Theorem 4.3 two maps $j(2\ell)$ and $j''(2\ell)$ are described, where j and j'' are integers ≥ 3 and $\not\equiv 2 \pmod 3$, and 2ℓ can be any even integer ≥ 4. These maps can be redefined as follows:

- if $2\ell \equiv 0, 4 \pmod 6$ then $j(2\ell) = 2\ell$;
- if $2\ell \equiv 2 \pmod 6$ then from the vertex 2ℓ there is a unique path going backwards in the graph along odd numbered vertices, $j(2\ell)$ then is the last odd number on this path;
- if $2\ell \equiv 0, 2 \pmod 6$ then $j''(2\ell) = \ell$;
- if $2\ell \equiv 4 \pmod 6$ then from the vertex ℓ there is a unique path going backwards in the graph along odd numbered vertices, $j''(2\ell)$ then is the last odd number on this path.

These definitions suffice to describe and analyze the arguments of page 8–11.

The map j is bijective, the map j'' is not. The *annihilation graph* introduced on page 11, defined according to the action of Algorithm 4.9 on 2ℓ, connects every even integer 2ℓ to $j^{-1}(j''(2\ell))$. Walking a path in this graph means iterating the map $j^{-1} \circ j''$.

The equations $\eta_{j_i} - \eta_{j_k} = 0$ in Lemma 4.11 can be reformulated as the statement that j_i and j_k stay in the same connected component of the $3n+1$ graph. Clearly all walks implicitly described in [O] take place inside the connected component of the graph that one started in, because for every edge in the annihilation graph connecting 2ℓ and $j^{-1}(j''(2\ell))$, there is an undirected path in the $3n+1$ graph connecting the vertices 2ℓ and $j^{-1}(j''(2\ell))$. This implies that the connected components of the $3n+1$ graph correspond one to one to the connected components of the annihilation graph. Thus in order to prove the $3n+1$ conjecture it suffices to prove that the annihilation graph has only one connected component, namely the known one.

The argument on pages 10–11 describes that starting from the bottom of the annihilation graph one may extend it in the backward direction at every new vertex that is found. In other words, every vertex has at least one predecessor. This argument, albeit being correct on its own, however is completely consistent with the possibility that the annihilation graph has more connected components. The argument then simply shows that in each such component every vertex has at least one predecessor. I see in [O] no convincing argument why such components other than the known one cannot exist. In particular, the "crucial assumption" in Theorem 4.12, that, "starting the algorithm with $2\ell_0$, there will always be a value of the 2ℓ column which is smaller than $2\ell_0$", is not proven, and the statement "The set of all vertices $(2n, l)$ in all levels will contain all even numbers $2n \geq 6$ exactly once" on page 11 (10 lines from the bottom) is not proven when it is taken to mean that all those vertices appear in the one known connected component. Thus I fail to see that the arguments presented in [O] suffice to prove the $3n+1$ conjecture.

Further comments:

- It seems that the only purpose of the discussion of functional equations and linear operators is to introduce the maps j, j'' as described above. As we saw, these maps can easily be defined as walks in the $3n+1$ graph, without the entire background based on the work of Berg and Meinardus [BM]. Once this becomes clear, this particular choice of walks becomes arbitrary; one could define similar walks in many more ways, leading to "annihilation graphs" similar to the one in [O], for which similar conclusions can (not) be drawn.
- For the $3n-1$ problem (which is the same as the $3n+1$ problem on the negative integers) three cycles exist, i.e. the $3n-1$ graph has three separate connected components. The arguments of [O] can probably be adapted to this case, providing a good example of the problem in the proof.
- On page 4, note that $\ker(U) = \langle 1, F(z), F(z^3), F(z^5), F(z^7), \ldots \rangle$, where $F(z) = z + z^2 + z^4 + z^8 + z^{16} + \ldots$ is the so called *Fredholm series*. Clearly $\dim \ker(U) = \infty$. Further, note that $\ker(V) = \langle 1, t_1, t_3, t_4, t_6, \ldots \rangle$, where t_i are the polynomials defined by Algorithm 3.1. Clearly $\dim \ker(V) = \infty$. Both results can be found in [BM].
- In several places the paper confuses the reader by statements of the structure "If P then Q", where the author actually means "If P then Q, and since P is true, so is Q". A good example is Conclusion 4.18, where it says "$U[h] = 0$ for $h \in \mathcal{K}_V$ implies $\mathcal{K} = \Delta_2$", while the author probably means to say that also $U[h] = 0$ is true, and hence so is $\mathcal{K} = \Delta_2$.

On June 8, Prof. Opfer writes to me: "It is true that (in the very end) some arguments are missing."

References

[BM] Lothar Berg and Günter Meinardus, *Functional equations connected with the Collatz problem*, Results in Mathematics **25** [1994], 1–12.

[O] Gerhard Opfer, *An Analytic Approach to the Collatz* $3n+1$ *Problem*, Hamburger Beiträge zur Angewandten Mathematik Nr. 2011-09, May 2011.